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Original Contribution

A learning model for oscillatory networks

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Abstract

A learning model for coupled oscillators is proposed. The proposed learning rule takes a simple form by which the intrinsic frequencies of the component oscillators and the coupling strength between them are changed according to the effects of the input signals on the dynamics of the oscillator. In learning mode, each component oscillator receives a teacher signal of desired phase and frequency, and a desired parameter set for generating the desired pattern is acquired by the proposed learning rule. It is known that the basic locomotor patterns of many living bodies are generated by coupled neural oscillators. The proposed learning rule could be a learning model used by such neural systems to acquire an adequate parameter set for generating a desired locomotor pattern.

Key words

learning method, coupled oscillators, central pattern generator

1 Introduction

The basic locomotor patterns of most living bodies, such as swimming and walking, are generated by the central pattern generator (CPG), which is a network of neurons situated in the spinal cord of vertebrates and the segmental ganglia of invertebrates. It is known that the CPG is composed of collective neural oscillators, which individually provide the signals controlling the movement of each limb or movement of the body (Pearson, 1976, Grillner, 1985, Pearce and Friesen, 1988). The phase relation between these component oscillators must be well organized so as to generate an adequate locomotor pattern, such as the gating or the undulation of a body. Although the CPG receives the control signals from the higher motor center (Rovainen, 1967, Rovainen, 1974), the isolated CPG itself can generate an adequate firing pattern to produce a basic locomotor pattern (Wallen and Williams, 1984). This fact suggests that adequate couplings and internal frequencies of the oscillators exist within the CPG.

Although some theoretical works have discussed the mechanism of the generation of locomotor patterns by the CPG (Cohen et al., 1982, Bay and Hemami, 1987, Williams et al., 1990, Schöner et al., 1990, Collins and Stewart, 1992,1993a,b), few learning models have been proposed for the acquisition of an adequate parameter set of the CPG. Doya and Yoshizawa (1992) showed an adaptive learning method of the coupling strength between the CPG and a physical system to acquire a desired locomotor pattern. Ermentrout and Kopell (1994) proposed a learning rule to acquire an instructed desired coupling weight between nonlinear oscillators with nearest neighbor coupling so as to generate an instructed desired phase pattern. In these models, the intrinsic frequencies of component oscillators are fixed and the proposed learning rule can be applied only for a specific neural oscillator. In this article, we propose a basic learning rule of the coupling strengths and the intrinsic frequencies for the simple phase oscillators with multiple coupling of which dynamics is a typical form derived from some classes of nonlinear oscillators. The proposed learning rule takes a simple form which depends on the effect of input signals on the phase dynamics. By this rule, the coupled oscillators learn an adequate phase pattern from the teacher signal.

2 Learning rule: Forcing oscillation method

Insert figure 1 around here.

In this paper, we consider the case that each component oscillator among collective coupled oscillators receives a forcing oscillation of a desired phase and frequency as a teacher signal in the learning phase. Such a case corresponds to individual component oscillators in the CPG receiving the teacher signal from higher motor centers. Suppose that no recurrent connection exists between component oscillators and that there are no isolated oscillators. The oscillators are numbered such that there are no connections from oscillator j to oscillator i for the oscillator indexes $i < j$. The dynamics between coupled oscillators is assumed to take the following form (Fig.1).

$$\begin{cases} \dot{\theta}_i = \omega_i + R_i + \epsilon_f F_i, \\ \dot{\tilde{\theta}}_i = \Omega, \quad (i = 1, \ldots, N), \end{cases} \tag{1}$$

$$R_i \equiv \begin{cases} 0 & (i = 1) \\ \displaystyle\sum_{j \in J_i} \sum_{l=1}^{L_{ij}} w_{ij}^l R_{ij}^l, & (i = 2, \ldots, N), \end{cases} \qquad R_{ij}^l \equiv R(\phi_{ij} - \psi_{ij}^l), \quad F_i \equiv F(\tilde{\phi}_i),$$

where θ_i, $\tilde{\theta}_i \in \mathbf{S}^1$, $(i = 1, \ldots, N)$ are the phases of the component oscillator and the

[1]Here, we define $\mathbf{S} = \mathbf{R}$, (mod 1).

teacher signal, respectively, N is the number of the component oscillators, $\phi_{ij} \equiv \theta_j - \theta_i$ is the phase difference between the i-th oscillator and the j-th one, $\tilde{\phi}_i = \tilde{\theta}_i - \theta_i$ is the phase difference between the teacher signal and the component oscillator, J_i shows the ensemble of the oscillators sending signals to the i-th oscillator, and ω_i is the intrinsic frequency of the i-th oscillator. The frequencies of the teacher signals are assumed to have the same value Ω. Supposing various different phase-delayed couplings between oscillators, w_{ij}^l and ψ_{ij}^l ($l = 1, \ldots, L_{ij}$) show the coupling strength and the phase delay of the l-th coupling from the j-th oscillator to the i-th one. We assume that $\sum_{j \in J_i} \sum_{l=1}^{L_{ij}} w_{ij}^l = o(1)$, i.e., the effect from the coupled oscillator is less than the effect of the intrinsic dynamics, where L_{ij} shows the number of the different phase-delayed couplings from the j-th oscillator to the i-th one. When each oscillator is composed of some neural cells, such various phase delayed couplings between oscillators mean that the different cells in a component oscillator send (or receive) signals to (or from) the other oscillator. $R(\phi)$ and $F(\phi)$ are the C^1 periodic functions showing the effects of signals from the coupled oscillator and the teacher signal, respectively, which are assumed to be functions of the phase difference between oscillators. $\epsilon_f \ll 1$ is the small constant value indicating the strength of the teacher signal. The above phase dynamics can be derived from many nonlinear oscillators by lowest approximation when the attraction to the limit cycle is sufficiently strong and the interactions between oscillators are small (Ermentrout and Kopell, 1991).

Insert figure 2 around here.

The purpose of the learning is to obtain a desired parameter set, i.e., the coupling weight w_{ij} and the intrinsic frequency ω_i, which together enable the coupled oscillators to generate the same phase pattern as the teacher signal. For this purpose we propose the following learning rule (Fig.2),

$$\begin{cases} \dot{\omega}_i &= \varepsilon(\epsilon_f F_i + R_i) \\ \dot{w}^l_{ij} &= \varepsilon\gamma \cdot \epsilon_f F_i \cdot R^l_{ij}, \end{cases} \tag{2}$$

where $0 < \varepsilon, \gamma \ll 1$ are the constants determining the learning velocity. This learning rule implies that the intrinsic frequency, as it changes according to the total effect of the input signals, approaches the current frequency. The second rule implies that the coupling strength changes according to the correlation between the effects of the signal from the coupled oscillator and the teacher signal, i.e., when both signals cause a phase shift in the same direction, the coupling weight is enforced. If the above learning rule successfully converges, $F_i = 0$ and $R_i = 0$ are obtained, which implies that a phase relation equivalent to that of the teacher signal is acquired and the learned point is an equilibrium point after removing the instruction of the teacher signal. Concerning this learning rule, the following theorem is obtained.

Theorem 1 *When the parameters of the dynamics (1) change according to the learning rule (2), $\omega_i = \Omega$ and $\tilde{\phi}_i = 0$ $(i = 1, \ldots, N)$ are obtained, except in the case where the state stays at the unstable points, and if the following conditions are satisfied.*

1. *The learning velocity is sufficiently slow and a stable solution always exists in the phase difference space $\tilde{\boldsymbol{\phi}} = (\tilde{\phi}_1, \ldots, \tilde{\phi}_N)$, i.e., $\dot{\tilde{\phi}}_i \simeq 0$ is satisfied during the learning.*

2. *All oscillators except oscillator 1 receive more than two phase-delayed signals from an oscillator, which give different zero points of the interaction effect $R(\phi - \psi^l)$. That is,*

$$\forall i = 2, \ldots, N,\ \exists j \in J_i,\ \forall \phi \in \mathbf{S}, \quad \sum_{l=1}^{L_{ij}} (R(\phi - \psi^l_{ij}))^2 \neq 0$$

is satisfied.

3. *A C^1 function $h : \mathbf{S} \to \mathbf{S}$ which satisfies $h(0) = 0$, $\forall\phi \in \mathbf{S}$, $h'(\phi) > 0$ and $\forall\phi \neq 0$, $h(\phi) \neq 0$ exists such that $F(\phi)$ and $\sin 2\pi h(\phi)$ have the same sign in $\phi \in \mathbf{S}$; in other words, the dynamics $\dot{\phi} = -F(\phi)$ is stable only at $\phi = 0$.*

The first condition also requires that the initial intrinsic frequencies must satisfy $\omega_i \simeq \Omega$ for the existence of the phase-locked solution.

If the learning is successfully done, the effect of the teacher signal F_i becomes zero in eq.(1). Therefore it is expected that a phase relation equivalent to that of the teacher signal would be recalled after removing the teacher signal if its stability is guaranteed. About the stability of the learned phase pattern the next theorem holds (see appendix for the proofs of these theorems).

Theorem 2 *Assume that $\omega_i = \Omega$, $\tilde{\phi}_i = 0$, ($\phi_{ij} = \phi^d_{ij}$, $\phi^d_{ij} \equiv \tilde{\theta}_j - \tilde{\theta}_i$), ($i = 1, \ldots, N$) are obtained in the dynamics (1) by the learning rule (2). The learned phase pattern with the learned frequency is stable in the dynamics (1) without the term about the teacher signal F_i, if the following relation is satisfied at the learned point,*

$$\left| \sum_{j\in J_i} \sum_{l=1}^{L_{ij}} w^l_{ij} R'(\phi^d_{ij} - \psi^l_{ij}) \right| > \epsilon_f F'(0), \tag{3}$$

i.e., the effect of the teacher signal on the dynamics of the oscillator is less than the sum of the effects of the signals from the coupled oscillators.

Remark To recall the learned phase relation from the random initial phase values, the global stability of the learned point must be guaranteed, and this depends on the function R.

For instance, if a network with the function $R(\phi) = \sin 2\pi\phi$ and $(\psi^1_{ij}, \psi^2_{ij}) = (0, 1/4)$ satisfies the condition in theorem 2 by a weight set at the learned point, the global stability of the point is guaranteed.

In the above discussions, the effects of the input signals to an oscillator are given by the functions of the phase difference between oscillators. In many cases, expressing these functions in more general forms, such as $R(\theta_i, \theta_j)$ and $F(\theta_i, \tilde{\theta}_i)$, would better approximate a model of coupled neural oscillators. Although no stable phase-locked solution exists in two oscillators except a 1:1 frequency ratio when the effect of the coupling is a function of the phase difference, the coupled nonlinear oscillators with more general coupling terms show many stable solutions for various frequency ratios. (The stable frequency locking depends on the coupling term; see Hoppensteadt and Keener (1982)). Now we extend the learning rule (2) to the form that enables the learning of a desired phase pattern instructed by various frequency ratios.

In this case the coupling term in the dynamics of oscillators (1) is replaced by the form,

$$R^l_{ij} \equiv R(\theta_i, \theta_j - \psi^l_{ij}), \quad F_i \equiv F(\theta_i, \tilde{\theta}_i). \tag{4}$$

Suppose that the teacher signal with the frequency ratio $\Omega_1 : \Omega_2 : \cdots = n_1 : n_2 : \cdots$, ($n_1$, n_2 : natural numbers) is given to the coupled oscillators, and we have $\omega_i \simeq \Omega_i$. By setting $\Omega \equiv \Omega_i / n_i$ and replacing ω_i, θ_i, $\tilde{\theta}_i$ by $n_i\omega_i$, $n_i\theta_i$, $n_i\tilde{\theta}_i$, respectively, the following dynamics is obtained from the dynamics (1) with eq.(4) by applying the averaging theory (Ermentrout and Kopell, 1991):

$$\begin{cases} \dot{\theta}_i = \omega_i + \bar{R}_i + \epsilon_f \bar{F}_i, \\ \dot{\tilde{\theta}}_i = \Omega, \quad (i = 1, \ldots, N) \end{cases} \tag{5}$$

$$\bar{R}_i \equiv \sum_{j\in J_i}\sum_{l=1}^{L_{ij}} w_{ij}^l \bar{R}_{ij}^l, \quad \bar{R}_{ij}^l \equiv \bar{R}(\phi_{ij} - \psi_{ij}^l), \quad \bar{F}_i \equiv \bar{F}(\tilde{\phi}_i),$$

where $\bar{R}$ and $\bar{F}$ show the time-averaged functions of R and F, respectively, and are equivalent to the form,

$$\bar{R}(\phi) = \int_0^1 R(\theta, \theta+\phi)d\theta, \quad \bar{F}(\phi) = \int_0^1 F(\theta, \theta+\phi)d\theta. \tag{6}$$

Therefore, the learning rule (2) can be applied by using the time-averaged term, because the dynamics (5) takes the same form as eq.(1), that is, we obtain the following learning rule,

$$\begin{cases} \dot{\omega}_i &= \varepsilon(\epsilon_f \bar{F}_i + \bar{R}_i) \\ \dot{w}_{ij}^l &= \varepsilon\gamma \cdot \epsilon_f \bar{F}_i \cdot \bar{R}_{ij}^l. \end{cases} \tag{7}$$

According to the learning rule for the coupling weight in eqs. (2) and (7), the weights might become too large in some situations because there are no restrictions on the value. From a physiological viewpoint, it has been reported that the weight value saturates for the repeated stimulus (Baxter and Byrne, 1993). By assuming the weight w_{ij}^l as a function of a variable s_{ij}^l, an adequate value for s_{ij}^l can be learned in the same manner as for the above learning rule for the weight w_{ij}^l; that is, the learning rule,

$$\begin{aligned} w_{ij}^l &= W(s_{ij}^l), \\ \dot{s}_{ij}^l &= \varepsilon\gamma \cdot \epsilon_f F_i \cdot R_{ij}^l, \end{aligned}$$

also enables the learning and recalling of a teacher signal, if the C^1 function $W : \mathbf{R} \to \mathbf{R}$ satisfies $\forall s \in \mathbf{R}$, $W'(s) > 0$ (the proof is not shown here but obtained in the same manner as theorem 1). If the sigmoid function is taken as $W(s)$, the value of the coupling weight can be restricted.

3 Simulation results

3.1 Learning a phase difference between oscillators

First, a simulation to learn a phase difference between oscillators was performed. The dynamics of oscillators and the learning rule follow eq. (1) and eq. (2), respectively, and the functions take the form,

$$R(\phi) = \sin 2\pi\phi, \qquad F(\phi) = \sin 2\pi\phi. \tag{8}$$

Unidirectional coupling from the oscillator 1 to the oscillator 2 was assumed, and the number of phase-delayed couplings was set as two. Parameters are shown in the caption for Fig.3.

Not only when the oscillators synchronize each other but also when the frequencies of oscillators are completely different and no frequency locking occurs in the initial state, the phases of oscillators always become in phase with the instructed teacher signals from random initial weight values. The acquired phase relation was always recalled stably when the condition in theorem 2 held, but sometimes wrongly recalled when the condition was broken. Figures 3 and 4 show the error and the phases of oscillators during and after learning, respectively, where the phases are expressed by $y_i = \cos 2\pi\theta_i$ and $\tilde{y}_i = \cos 2\pi\tilde{\theta}_i$. Although frequency locking did not occur in the initial state, because the initial frequencies were set as $(\omega_1(0), \omega_2(0)) = (0.5, 3.0)$ Hz, the phase relation converged to the desired one within 10 s, and the learned phase relation was recalled after learning.

3.2 Learning the 2:1 frequency ratio phase pattern

Second, a simulation result to learn the 2:1 frequency ratio phase pattern in two coupled oscillators is shown.

Insert figure 5 around here.

Insert figure 6 around here.

Taking the effect of the input signal as the product of the input signal $Q(\theta)$ and the function $P(\theta)$ which shows the phase sensitivity of the oscillator, the interaction terms in eq.(4) take the forms

$$R^l_{ij} \equiv P(\theta_i)Q(\theta_j - \psi^l_{ij}), \quad F_i \equiv P(\theta_i)Q(\tilde{\theta}_i).$$

The above formulation is obtained as an approximated form of a nonlinear oscillator when the dynamics of the oscillator can be transformed to the normal form for Andronov-Hopf bifurcation (Nishii et al., 1994).

In this simulation, the functions are given by

$$P(\theta) = -(\sin 2\pi\theta + \sin 4\pi\theta)/2, \quad Q(\theta) = \cos 2\pi\theta. \tag{9}$$

The function P is composed of two frequency components in order to make it possible that the oscillators have a stable phase locked solution with a $2:1$ frequency ratio. The unidirectional coupling from the oscillator 1 to the oscillator 2 is assumed, and two different phase-delayed couplings are prepared. The learning is done according to eq.(7). The time averaged term in eq.(7) is given by the first-order low-pass filter:

$$\tau \dot{\bar{R}}^l_{ij} = -\bar{R}^l_{ij} + R^l_{ij}, \quad \tau \dot{\bar{F}}_i = -\bar{F}_i + F_i, \tag{10}$$

where $\tau = 3.0$ is a time constant. When each initial frequency of the oscillator was almost the same as its teacher signal and frequency locking occurred between them, each phase of the oscillator always became in phase with its teacher signal by learning. The acquired phase relation was recalled successfully when the condition in theorem 2 was satisfied.

We next provide a sample result when the initial condition does not satisfy the first condition in theorem 1. In this simulation, the initial intrinsic frequencies of the oscillators and the teacher signals are set as $(\omega_1, \omega_2) = (1.8, 0.6)$ and $(\Omega_1, \Omega_2) = (1.4, 0.7)$ [Hz], respectively; therefore, no frequency-locked solution exists between oscillators before learning. Figure 5 indicates the error during learning, and Fig.6 shows the output signals of the oscillators and the phase difference between oscillators defined by $\phi \equiv \theta_1 - 2\theta_2$ in the learning and recalling modes. It was shown that the oscillators become in phase with the teacher signals and that, in the recalling mode, the learned phase pattern is recalled stably from random initial phases. When, in the initial stages, the oscillators were trapped by frequency locking of an undesired ratio, the learning failed. For instance, an early $2:1$ frequency locking between oscillator 1 and teacher signal 1 makes an escape to desired frequency ratio very difficult.

The results from this and the prior simulation suggest that an appropriate frequency locking solution is not mandatory for acquiring the desired phase relation by rules (2) and (7) but it is important that the oscillators not be trapped in an inappropriate frequency locking.

3.3 Learning the lamprey-like phase pattern

The leech, the lamprey, and numerous fish swim by propagating the metachronal wave along their bodies. In these swimming patterns, the wavelength of a metachronal wave is almost one body length for all swimming velocities, and the phase delay is nearly identical between neighboring segments (Grillner et al., 1988). Below is a simulation of the CPG learning of such a phase pattern among coupled oscillators.

Insert figure 7 around here.

Assuming an all-to-all coupling between oscillators, the dynamics between oscillators can be given by

$$\begin{cases} \dot{\theta}_i &= \omega_i + \sum_{j\neq i}\sum_{l=1}^{L} w_{ij}^l P(\theta_i)Q(\theta_j - \psi^l) + \epsilon_f P(\theta_i)Q_f(\tilde{\theta}_i) \\ \dot{\tilde{\theta}}_i &= \Omega, \quad (i = 1, \ldots, N), \end{cases} \tag{11}$$

where $P(\theta)$ shows the phase sensitivity, $Q(\theta)$ is the output signal from an oscillator, $Q_f(\tilde{\theta})$ is the teacher signal, and the three functions take the following forms

$$P(\theta) = -\sin 2\pi\theta, \quad Q(\theta) = \cos 2\pi\theta, \quad Q_f(\tilde{\theta}) = \cos 2\pi\tilde{\theta}. \tag{12}$$

The initial phase of the teacher signal is set as $\tilde{\theta}_i(0) = -i/N$ $(i = 1, \ldots, N)$ so as to equalize the phase differences between neighboring oscillators. The numbers of oscillators and phase-delayed connections are set as $N = 20$ and $L = 2$, respectively. The coupling weight is expressed as a function of the variable s_{ij}^l (eq.(8)). The function $W(s)$ takes the sigmoid form, $w_{ij}^l = w_{max}\{2/(1.0 + \exp(-s_{ij}^l/a)) - 1.0\}$, and the initial value of s_{ij}^l is randomly distributed from -0.1 to 0.1. The learning rule is given by eqs.(7) and (8) The time-averaged term is calculated just as in the previous simulation. For all of the frequencies of the teacher signals, $\Omega = 1.0$ [Hz], and the initial intrinsic frequencies of the oscillators (ω_i) are randomly set from 0.7 to 1.3 [Hz].

Figures 7 and 8 show the phase pattern and time course of the frequencies of the component oscillators in the simulation. In Fig.7 dots indicate the point where $\theta_i = 0$.

During learning, the frequencies of the oscillators converge to the same value as the frequency of the teacher signal, and the same desired pattern is obtained. After learning, the learned phase pattern is regenerated from random initial phases. Although the validity of the proposed learning rule for oscillators with all-to-all coupling has previously been considered too analytically difficult to demonstrate, the present simulation effectively confirms it. Concerning coupling weights, they are not uniquely determined by a desired phase relation because of their redundancy, therefore they do not converge to certain values by learning and their final form depends on their initial values.

This simulation can be used as a model to show that the CPG learns the appropriate firing pattern by receiving a teacher signal from the higher motor center, and that, after the learning is completed, the CPG itself can generate the adequate firing pattern without any control from a higher center.

4 Discussion

A learning rule for the oscillatory network was proposed and its validity was confirmed by computer simulations. The proposed learning rule takes a simple form which depends on the effects of input signals. Since each component oscillator is entrained by a teacher signal during learning, the coupled oscillators can generate a phase pattern similar to the teacher signal even before acquiring an adequate parameter set. After the learning is completed, the coupled oscillators themselves can generate the learned phase pattern, just as the CPG itself can generate the desired firing pattern without signals from a higher motor center. Rovainen (1974) has shown physiologically that the CPG receives efferent signals from the higher center, so it is possible that a higher center sends a teacher signal to the CPG to tune the CPG's parameters. Thus the proposed learning rule could be a candidate of a learning rule in the CPG if the higher center calculates the desired phase pattern and sends it to the CPG as a teacher signal.

Although we have proposed a learning rule for acquiring the instructed phase relation, many living bodies might learn adequate gait patterns without instructions. In future studies we must also consider the learning algorithm for obtaining the desired phase pattern without explicit teacher signals. For this case, we have shown by use of the evaluation function, showing the performance of the locomotion, that the similar learning rule enables acquisition of the desired motor command for the simple locomotor pattern (Nishii, 1995, 1997a).

In this article, a simple phase oscillator is treated as a component of the network. In a living system, a neural cell and a neural circuit work as an oscillatory unit in the CPG. To ascertain the validity of the proposed learning rule, we must consider its possible application in such neural oscillators (Nishii, 1997b), which would give us a new perspective on the mechanism of temporal-signal learning in the brain.

Appendix A Proof of Theorem 1

First, prove that $\omega_i = \Omega$ $(i = 1, \ldots, N)$ is obtained by the learning rule (2) under the conditions in the theorem. From eq.(1), the dynamics of the phase difference $\tilde{\phi}_i$ takes the form,

$$\dot{\tilde{\phi}}_i = \Omega - \omega_i - R_i - \epsilon_f F_i. \tag{A.1}$$

The proof is made under assumption that the above dynamics have a stable point and remain continuously in a steady state (condition 1) in the manner shown by Pineda (1987).

That is, the following relation is always satisfied.

$$0 \simeq \Omega - \omega_i - R_i - \epsilon_f F_i. \tag{A.2}$$

The stability condition at the point $\tilde{\boldsymbol{\phi}} \equiv \{\tilde{\phi}_1, \ldots, \tilde{\phi}_N\}$ is given by the eigenvalues of the linearized matrix Λ of the right side of eq.(A.1). Since the matrix Λ is the lower triangular matrix of which the diagonal terms are $-f_1, -(a_2 + f_2), \cdots$, and $-(a_N + f_N)$, where $a_i \equiv \sum_{j \in J_i} \sum_{l=1}^{L_{ij}} w_{ij}^l R'(\phi_{ij} - \psi_{ij}^l)$, $f_i \equiv \epsilon_f F'(\tilde{\phi}_i)$, $\phi_{ij} = \tilde{\phi}_i - \tilde{\phi}_j + \phi_{ij}^d$ and $\phi_{ij}^d \equiv \tilde{\theta}_j - \tilde{\theta}_i$, the phase relation obtained by eq.(A.1) must satisfy

$$f_1 > 0, \quad f_i + a_i > 0, \quad (i = 2, \ldots, N). \tag{A.3}$$

Consider the Liapunov function

$$V_1 = \frac{1}{2} \sum_{i=1}^{N} (\Omega - \omega_i)^2 . \tag{A.4}$$

Applying eqs.(2) and (A.2) gives

$$\dot{V}_1 = -\sum_{i=1}^{N} (\Omega - \omega_i)\dot{\omega}_i = -\varepsilon \sum_{i=1}^{N} (\Omega - \omega_i)^2 \leq 0.$$

Therefore, $\omega_i = \Omega$ is asymptotic stable and the following equation is obtained by the learning.

$$\epsilon_f F_i + R_i = 0. \tag{A.5}$$

The stable phase difference $\tilde{\phi}_1$ satisfies $F'(\tilde{\phi}_1) > 0$ from eq.(A.3). $F_1 = 0$ is obtained by the learning, since the oscillator 1 has no coupling from the other oscillator ($R_1 = 0$), and thus the phase difference $\tilde{\phi}_1 = 0$ can be obtained by condition 3.

Second, it is shown that the phase locked solution $\tilde{\phi}_i = 0$, $(i = 1, \ldots, N)$ is obtained by the learning. As the first step to this end we calculate $\dfrac{\partial \tilde{\phi}_i}{\partial w^l_{kj}}$. By the partial differentiation of the eq.(A.2) by w^l_{kj}, we obtain the following equation:

$$\begin{aligned} 0 &= -\delta_{ki} R^l_{kj} - \sum_{n \in J_i} \sum_{m=1}^{L_{in}} w^m_{in} \frac{\partial R(\phi_{in} - \psi^m_{in})}{\partial w^l_{kj}} - \epsilon_f F'_i \frac{\partial \tilde{\phi}_i}{\partial w^l_{kj}} \\ &= -\delta_{ki} R^l_{kj} - (a_i + f_i) \frac{\partial \tilde{\phi}_i}{\partial w^l_{kj}} + \sum_{n \in J_i} \sum_{m=1}^{L_{in}} w^m_{in} R^{m\prime}_{in} \frac{\partial \tilde{\phi}_n}{\partial w^l_{kj}}, \end{aligned}$$

where δ_{ji} is the Cronecker's delta. From the above equation, we have

$$\begin{pmatrix} a_1 + f_1 & & & 0 \\ & a_2 + f_2 & & \\ & & \ddots & \\ * & & & a_N + f_N \end{pmatrix} \begin{pmatrix} \dfrac{\partial \tilde{\phi}_1}{\partial w^l_{kj}} \\ \vdots \\ \dfrac{\partial \tilde{\phi}_N}{\partial w^l_{kj}} \end{pmatrix} = - \begin{pmatrix} 0 \\ \vdots \\ R^l_{kj} \\ 0 \\ \vdots \end{pmatrix} \begin{matrix} i = 1 \\ \vdots \\ i = k \\ \vdots \\ i = N. \end{matrix} \tag{A.6}$$

Hence, $\dfrac{\partial \tilde{\phi}_i}{\partial w^l_{kj}}$ for $i \leq k$ takes the form

$$\frac{\partial \tilde{\phi}_i}{\partial w^l_{kj}} = \begin{cases} -\dfrac{R^l_{ij}}{f_i + a_i} & (i = k) \\ 0 & (i < k). \end{cases}$$

Then, consider a Liapunov function which takes value zero only at $\tilde{\phi}_i = 0$:

$$V^i_2 = \frac{1}{\pi} \sin^2 \pi h(\tilde{\phi}_i), \quad (i = 1, \ldots, N). \tag{A.7}$$

Because the frequency learning is much faster than the weight learning $(\gamma \ll 1)$, we have

$\omega_i = \Omega$ during the weight learning. The time course of V_2 is given by the following forms,

$$\begin{aligned} \dot{V}_2^i &= h'(\tilde{\phi}_i)\sin 2\pi h(\tilde{\phi}_i)\sum_{k=1}^{N}\sum_{j\in J_k}\sum_{l=1}^{L_{kj}}\frac{\partial\tilde{\phi}_i}{\partial w_{kj}^l}\dot{w}_{kj}^l \\ &= \varepsilon\gamma h'(\tilde{\phi}_i)\sin 2\pi h(\tilde{\phi}_i)\sum_{k=1}^{n}\sum_{j\in J_k}\sum_{l=1}^{L_{kj}}\frac{\partial\tilde{\phi}_i}{\partial w_{kj}^l}\cdot\epsilon_f F_k\cdot R_{kj}^l, \end{aligned} \tag{A.8}$$

where we used eq.(2).

(i) From the result for V_1, we have $\tilde{\phi}_1 = 0$ ($V_2^1 = 0$) by the frequency learning.

(ii) Assume that $\tilde{\phi}_i = 0$ is established for $i < n$ by the learning. For $i = n$, after a long time, we obtain

$$\begin{aligned} \dot{V}_2^n &= \varepsilon\gamma h'(\tilde{\phi}_n)\sin 2\pi h(\tilde{\phi}_n)\sum_{j\in J_n}\sum_{l=1}^{L_{nj}}\frac{\partial\tilde{\phi}_n}{\partial w_{nj}^l}\cdot\epsilon_f F_n\cdot R_{nj}^l \\ &= -\varepsilon\gamma h'(\tilde{\phi}_n)\cdot\epsilon_f F_n\cdot\sin 2\pi h(\tilde{\phi}_n)\sum_{j\in J_n}\sum_{l=1}^{L_{nj}}\frac{(R_{nj}^l)^2}{f_n + a_n} \\ &\le 0, \end{aligned} \tag{A.9}$$

where we used conditions 2 and 3 and eq.(A.3). The equality is established at the point where $h(\tilde{\phi}_n) = 0, 1/2$. Since we have $V_2^i = 0$ only at $\tilde{\phi}_i = 0$, it is proved that $\tilde{\phi}_i = 0$, $(i = 1, \ldots, N)$ is asymptotic stable except at the unstable point satisfying $h(\tilde{\phi}_i) = 1/2$. The unstability of the point with $h(\tilde{\phi}_i) = 1/2$ is proven in the same way as the above by using the Liapunov function $V^i = 1/\pi \sin^2\pi\{h(\tilde{\phi}_i) - 1/2\}$. Remark that $R_i = 0$ is obtained from eq.(A.5) when $\tilde{\phi}_i = 0$ is acquired by the learning.

Appendix B Proof of Theorem 2

Here we consider the condition that instructed phase pattern by the teacher signal $\boldsymbol{\phi}^d \equiv \{\phi_1^d, \ldots, \phi_{N-1}^d\}$, $\phi_i^d \equiv \tilde{\theta}_{i+1} - \tilde{\theta}_i$ is stable in the phase difference space $\{\phi_1, \ldots, \phi_{N-1}\}$, $(\phi_i \equiv \theta_{i+1} - \theta_i)$ when the teacher signal is removed after learning.

Assuming $\omega_i = \Omega$ is established by the learning, the dynamics of the phase difference ϕ_i given by eq.(1) takes

$$\dot{\phi}_i = R_{i+1} - R_i + \epsilon_f F_{i+1} - \epsilon_f F_i. \tag{B.10}$$

Since $R_i = 0$ and $F_i = 0$ are obtained at the learned phase relation $(\tilde{\phi}_1, \boldsymbol{\phi}) = (0, \boldsymbol{\phi}^d)$, it is apparent that $\boldsymbol{\phi} = \boldsymbol{\phi}^d$ is an equilibrium point of the above dynamics even after the teacher signal F_i is removed. The stability condition at the desired phase relation $(\tilde{\phi}_1, \boldsymbol{\phi}) = (0, \boldsymbol{\phi}^d)$ in learning is given by the following form with eq.(A.3):

$$f^d > 0, \quad f^d + a_i^d > 0, \quad (i = 2, .., N), \quad a_i^d \equiv a_i|_{\boldsymbol{\phi}=\boldsymbol{\phi}^d}, \quad f^d \equiv \epsilon_f F'(0).$$

On the other hand, the stability condition at $\boldsymbol{\phi} = \boldsymbol{\phi}^d$ after learning is given by the eigen values of the linearized matrix of the right side of eq.(B.10) without the term F_i, that is, $a_i^d > 0$, $(i = 2, \ldots, N)$. If we have $f^d + a_i^d > 0$ at the learned point $(\tilde{\phi}_1, \boldsymbol{\phi}) = (0, \boldsymbol{\phi}^d)$ with the condition in the theorem in the learning mode, $a_i^d > 0$ is satisfied in the recalling state. Therefore, it is proven that the learned phase relation is stable if the condition in the theorem is satisfied.

Here, it is apparent that the recalled frequency is the same as the frequency of the teacher signal when the learned phase relation is recalled, since we have $F_i = 0$ and $R_i = 0$ at $\boldsymbol{\phi} = \boldsymbol{\phi}^d$ in dynamics (1).

Figure Legends

Fig.1: The coupled oscillators forced by teacher signals with desired frequency and phase. Each circle indicates an oscillator, and $\epsilon_f F_i$ indicates the effect of the teacher signal on each oscillator.

Fig.2: The proposed learning rule for coupled oscillators. The intrinsic frequency ω changes according to the sum of the effect of input signals, and the coupling strength w changes according to the correlation between the effects of the teacher signal F and the input signal R.

Fig.3: The time profile of the error in learning a phase difference. The error is given by the form $E = (\sum_{i=1}^{2} \sin^2 \pi\tilde{\phi}_i)/2$ in the learning phase and by $E = \sin^2 \pi(\phi - \phi^d)$ after learning, where $\phi \equiv \theta_1 - \theta_2$, $\phi^d \equiv \tilde{\theta}_1 - \tilde{\theta}_2$. The arrow indicates the time to stop the learning and to begin regenerating the learned phase pattern from random initial phases. The appropriate parameters are obtained within a few decade cycles, and fast recovery is obtained after learning. Parameters: $(\Omega_1, \Omega_2) = (1.0, 1.0)$, $(\psi^1, \psi^2) = (0, 0.2)$, $\epsilon_f = 0.5$, $\varepsilon = 0.5$, $\gamma = 1.0$. Initial values: $(\tilde{\theta}_1(0), \tilde{\theta}_2(0)) = (0.5, 0.7)$, $(\omega_1(0), \omega_2(0)) = (0.5, 3.0)$, $(\theta_1(0), \theta_2(0)) = (0.3, 0.0)$, $(w_{21}^1(0), w_{21}^2(0)) = (0.3, 0.3)$.

Fig.4: The time profile of the phase in learning a phase difference. The upper figure (teacher) shows the phases of the teacher signals for the oscillator 1 (solid line) and the oscillator 2 (dotted line), and the middle figure (output) shows the phases of the oscillator 1 (solid line) and the oscillator 2 (dotted line). The phase of an oscillator is plotted by the form $y_i = \cos 2\pi\theta_i$. The bottom figure (phase difference) shows the phase

difference between two oscillators $\phi \equiv \theta_1 - \theta_2$. The dotted line indicates the desired phase difference instructed by the teacher signals. (a)(b), learning mode (first 10 seconds and 10-20 seconds, respectively). (c), regenerating mode (20-30 seconds). The frequency and the phase of oscillators converge to the appropriate value within 20 seconds through learning. After learning, the learned phase pattern is regenerated within a few cycles.

Fig.5: The time profile of the error in learning a 2:1 frequency ration phase pattern. The error is given by the form $E = (\sum_{i=1}^{2} \sin^2 \pi\tilde{\phi}_i)/2$ in the learning phase and $E = \sin^2 \pi(\phi - \phi^d)$ after learning, where $\phi = \theta_1 - 2\theta_2$ and $\phi^d = \tilde{\theta}_1 - 2\tilde{\theta}_2$. The arrow indicates the time to stop the learning and begin to regenerate the learned phase pattern from random initial phases. The appropriate parameters are obtained within a few decade cycles, and fast recalling is obtained after learning. Parameters: $(\Omega_1, \Omega_2) = (1.4, 0.7)$. $(\psi^1, \psi^2) = (0, 0.2)$, $\epsilon_f = 0.5$, $\varepsilon = 0.5$, $\gamma = 1.0$. Initial values: $(\tilde{\theta}_1(0), \tilde{\theta}_2(0)) = (0.7, 0.8)$, $(\omega_1(0), \omega_2(0)) = (1.8, 0.6)$, $(\theta_1(0), \theta_2(0)) = (0.5, 0.0)$, $(w_{21}^1(0), w_{21}^2(0)) = (0.3, 0.3)$.

Fig.6: The time profile of the output signals of oscillators in learning a 2:1 frequency ration phase pattern. The upper figure (teacher) shows the teacher signal 1 ($Q(\tilde{\theta}_1)$, solid line) and the teacher signal 2 ($Q(\tilde{\theta}_2)$, dotted line), and the middle figure (output) shows the output signals of the oscillator 1 ($Q(\theta_1)$, solid line) and the oscillator 2 ($Q(\theta_2)$, dotted line). The bottom figure (phase difference) shows the phase difference between two oscillators $\phi \equiv \theta_1 - 2\theta_2$ (solid line) and the dotted line indicates the desired phase difference instructed by the teacher signals, $\tilde{\theta}_1 - 2\tilde{\theta}_2$. (a)(b), learning mode (first 10 seconds and 30-40 seconds, respectively) (c), regenerating mode (40-50 seconds). The frequencies and the phases of oscillators converge to the appropriate value within a few decade seconds through learning. After learning, the learned phase pattern is regenerated

from random initial phases within a few cycles.

Fig.7: The phase pattern of the coupled oscillators in learning of a lamprey-like firing pattern. (a), teacher signal; (b)(c), during learning (0-50[s]); (d)(e), after learning (recalling mode, 50[s]-). The abscissa shows the index of oscillators, and the dots show the time when the phase of each oscillator satisfies $\theta_i = 0$. Through learning the phase pattern approaches the teacher signal. At 50 [s], the learning is stopped and the memorized pattern is recalled from a random phase pattern. Parameters: $w_{max} = 0.5$, $a = 0.2$, $\epsilon_f = 0.5$, $\varepsilon = 0.1$, $(\psi^1, \psi^2) = (0, 0.25)$, $\gamma = 1.0$, $\tau = 3.0$ [s].

Fig.8: The time profile of the intrinsic frequencies of the oscillators in learning of a lamprey-like firing pattern. The frequencies converge to the desired value ($\Omega = 1.0$ Hz) instructed by the teacher signal.

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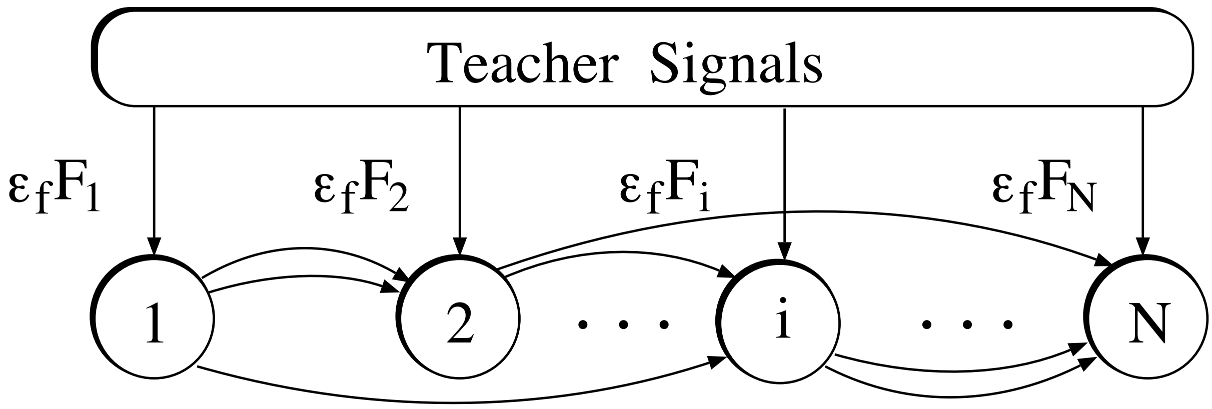

Teacher Signals
$\varepsilon_f F_1$
$\varepsilon_f F_2$
$\varepsilon_f F_i$
$\varepsilon_f F_N$
1
2
i
N

Fig.1:

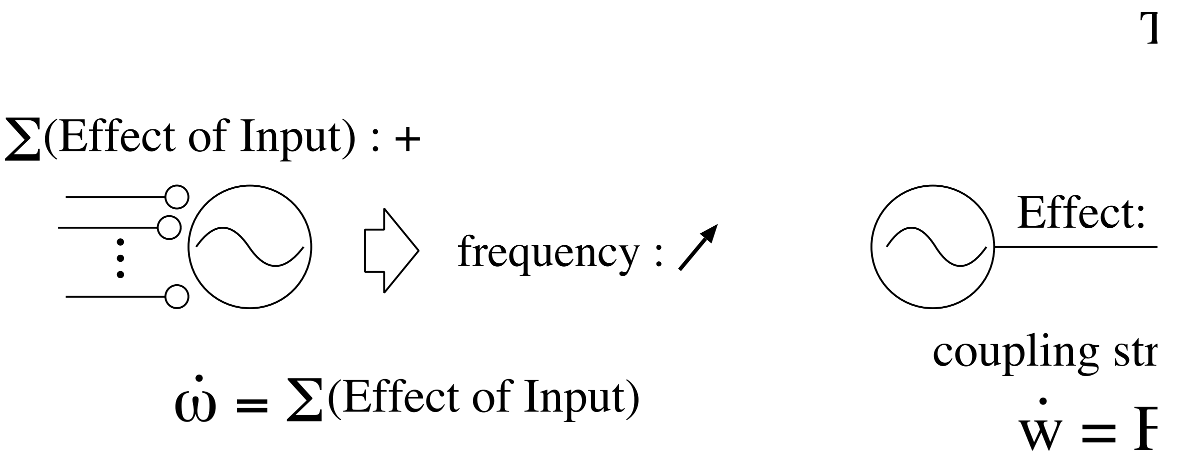
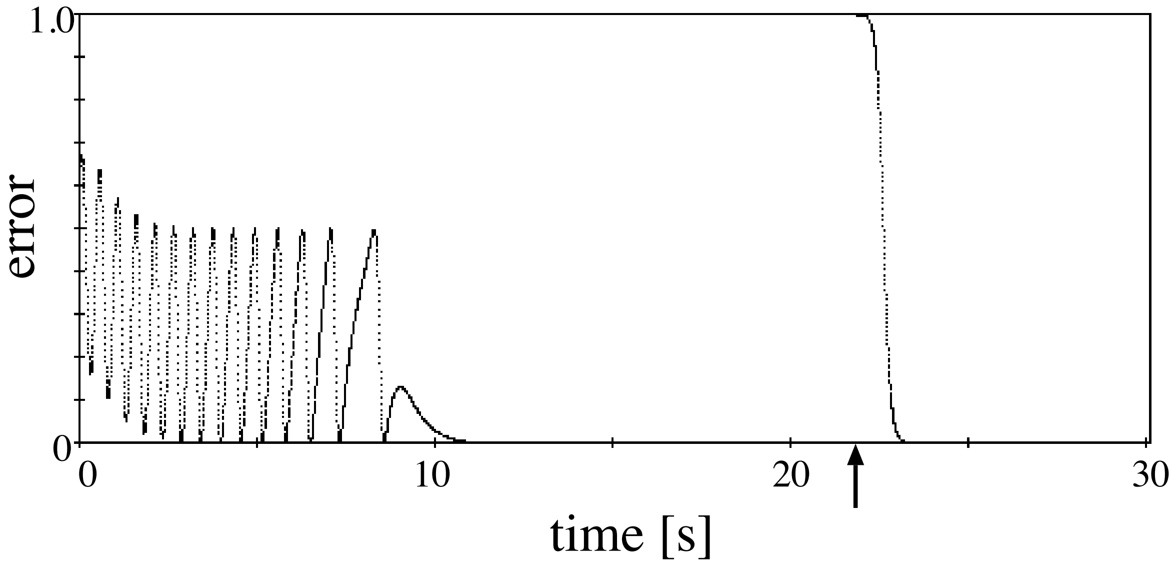

∑(Effect of Input) : +
frequency :
$\dot{\omega} = \sum$(Effect of Input)
Effect:
coupling str
$\dot{w} = F$

Fig.2:



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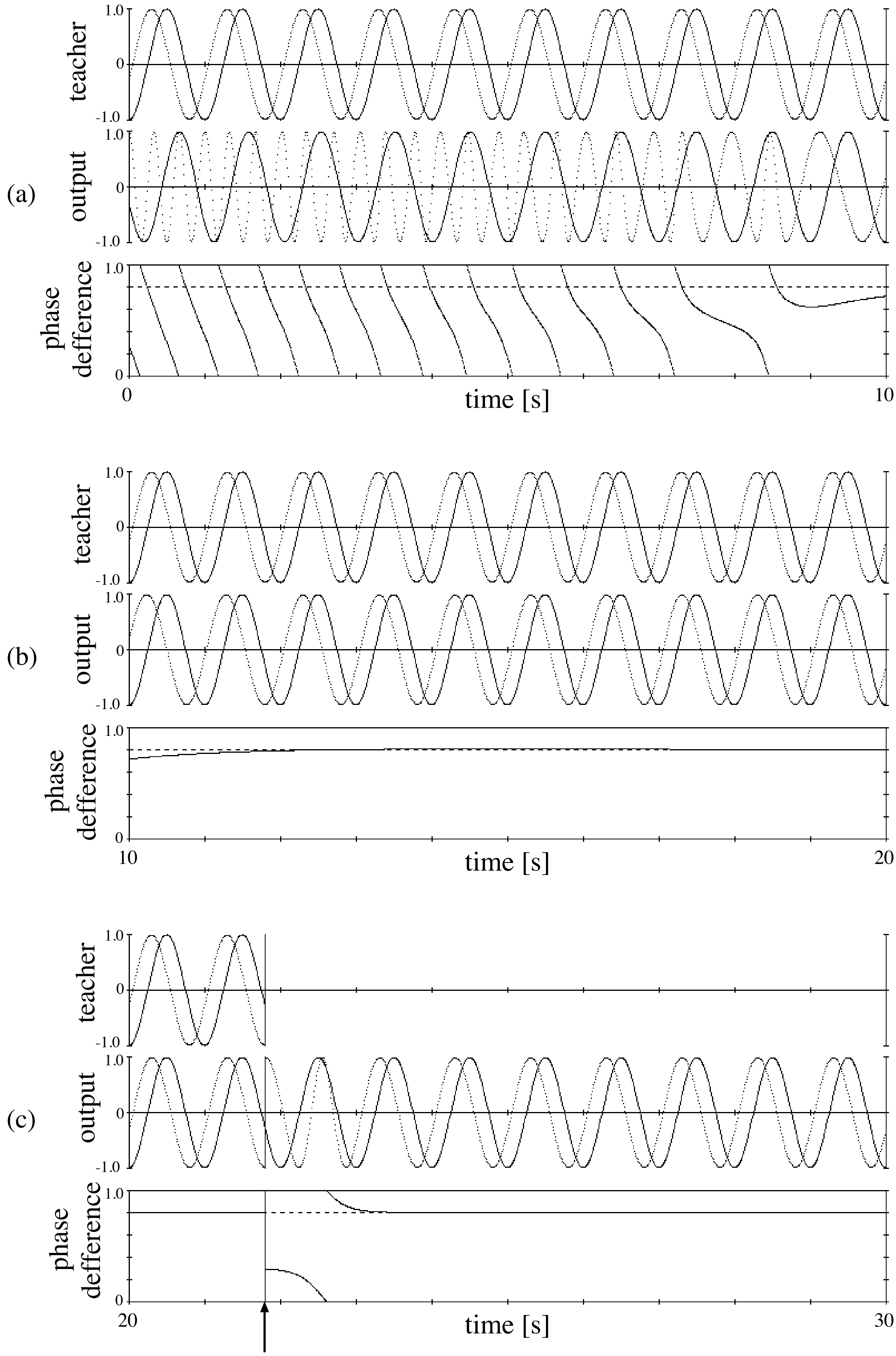

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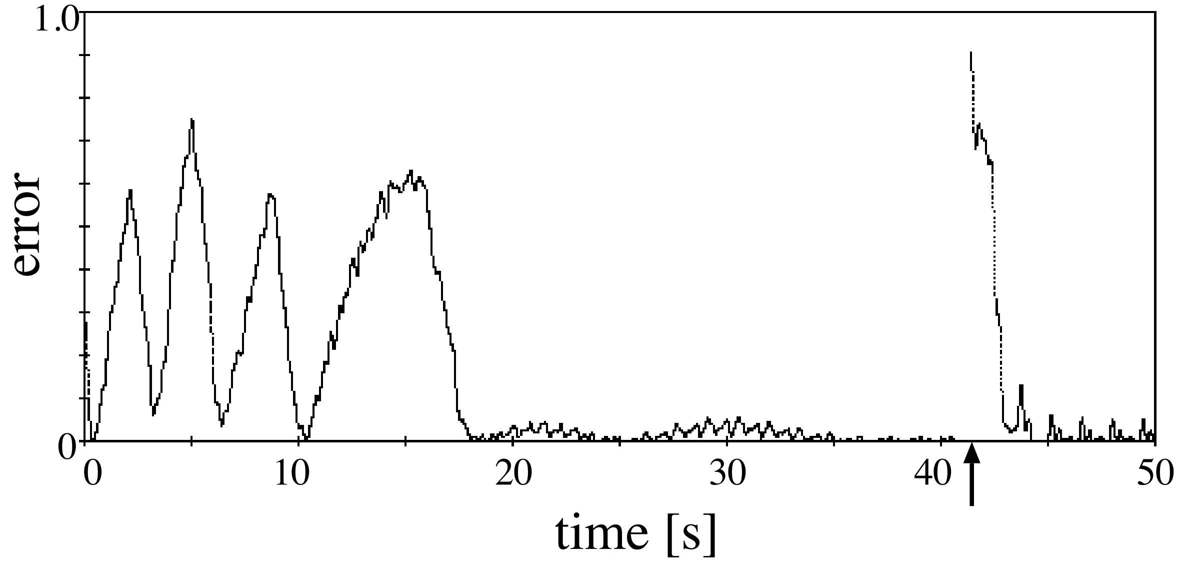

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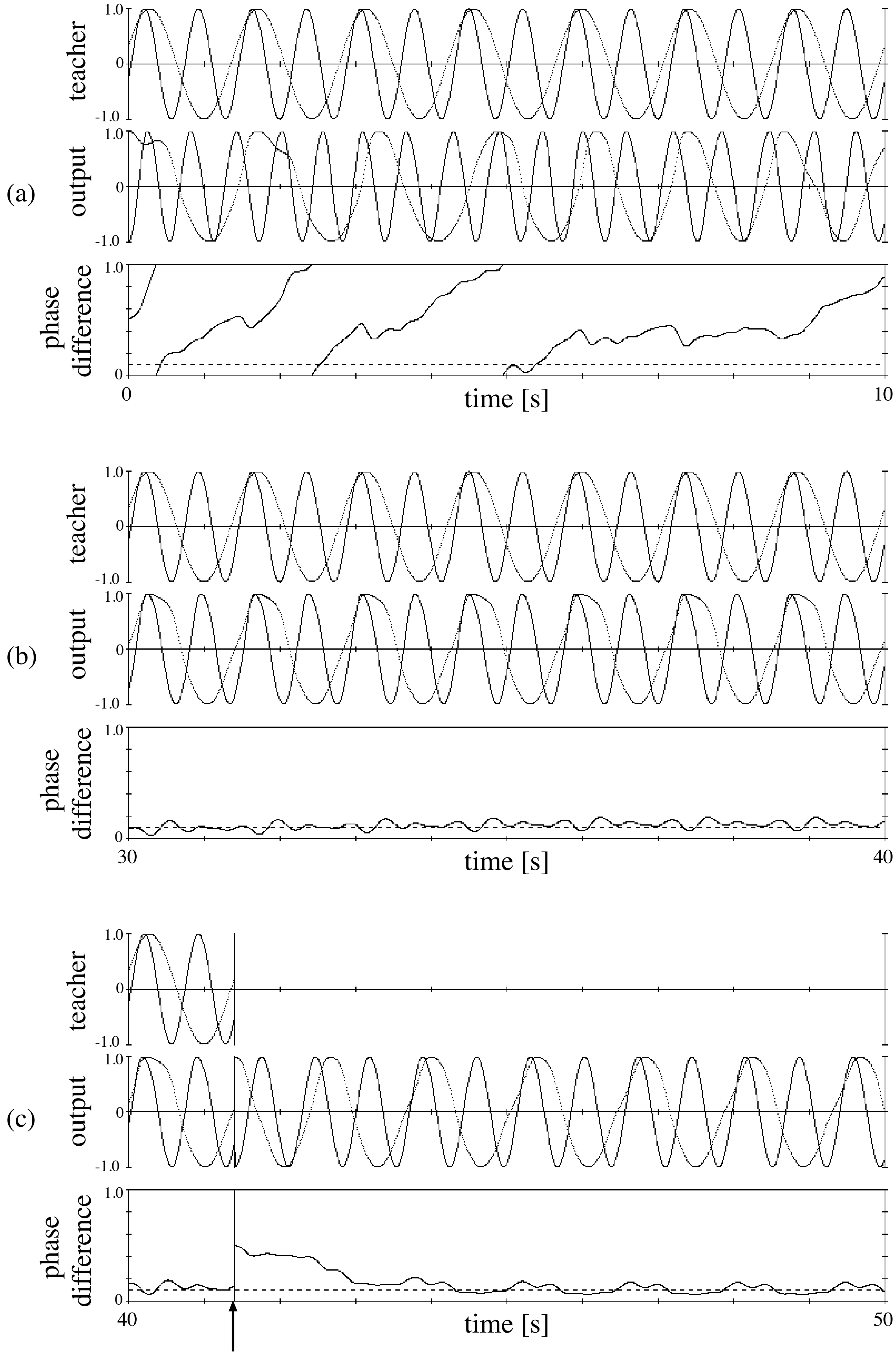

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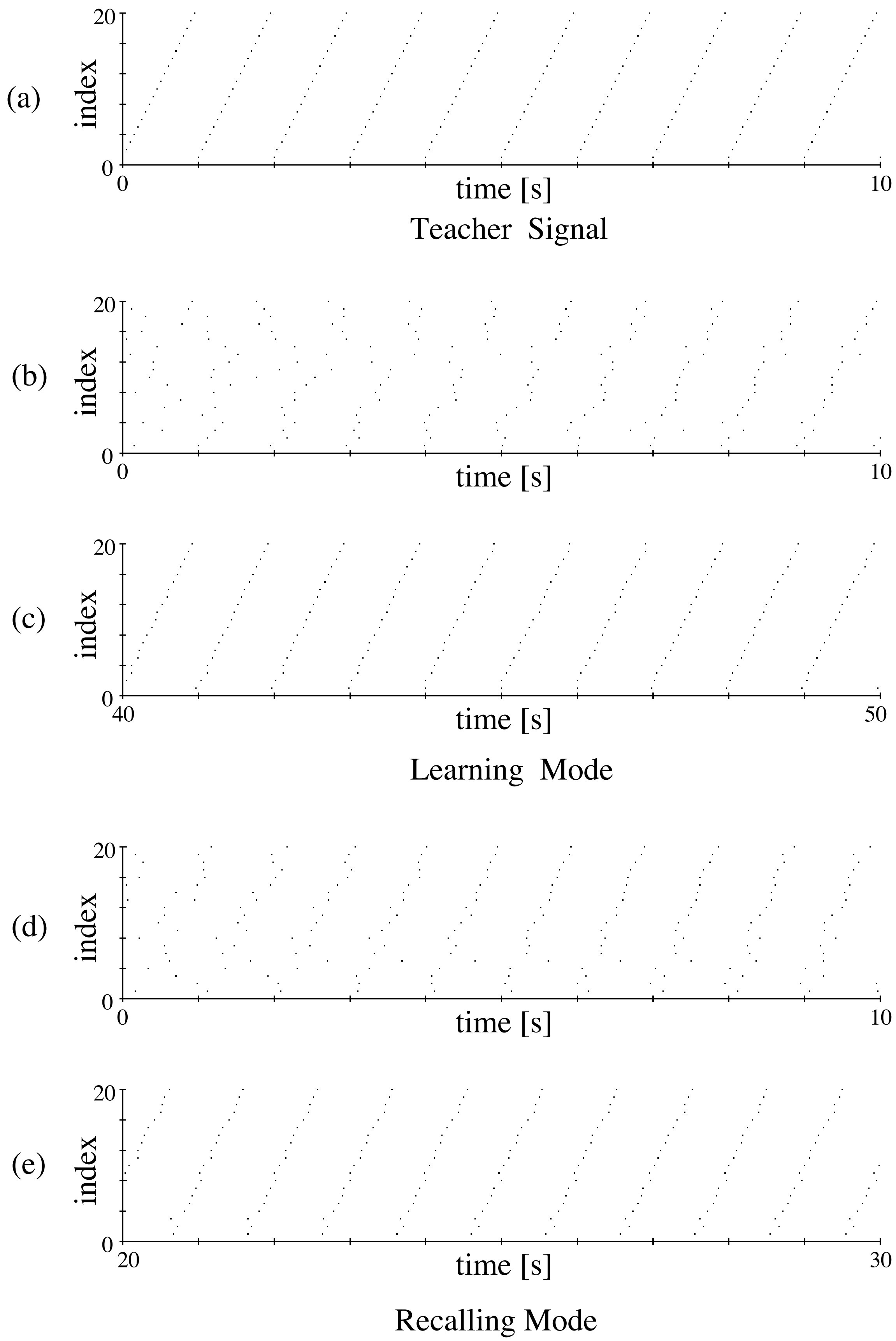

(a)
index
20
0
0
10
time [s]
Teacher Signal
(b)
index
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time [s]
(c)
index
20
0
40
50
time [s]
Learning Mode
(d)
index
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(e)
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time [s]
Recalling Mode

Fig.7:

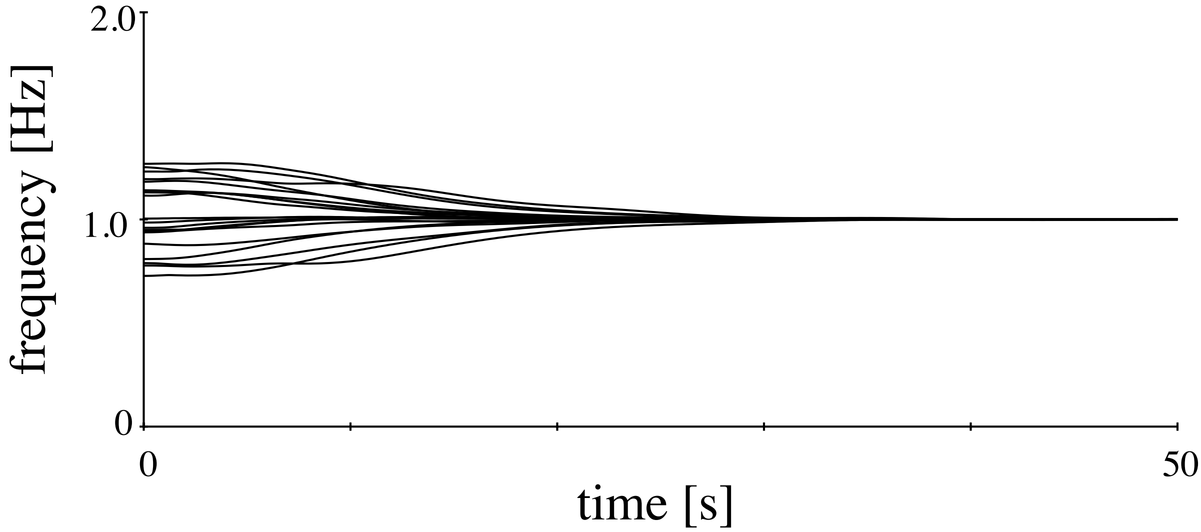

Fig.8: